THE EFFECTS OF GENERAL REGULAR TRANSFORMATIONS ON OSCILLATIONS OF SEQUENCES OF FUNCTIONS*

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1. Introduction

Recently the writer† has considered the behavior of continuous oscillation, continuous convergence, uniform oscillation, and uniform convergence of complex and real sequences of functions under complex and real regular transformations with triangular matrices; oscillation and convergence being in each case considered (1) over a set as a whole, (2) at a single point of a set, and (3) at all points of a set. It is the object of the present paper to outline an extension of that investigation, considering regular transformations of a general form which includes practically all of the transformations used in the theory of summability.‡

2. Transformations

Let T and A be sets of metric spaces, let T have a limit point t_0 not belonging to T, and let functions $a_k(t)$, $k=1, 2, 3, \cdots$, be defined over T. If a sequence $\{s_n(x)\}$, defined over A, is such that

$$(G) \qquad \sigma(t, x) = \sum_{k=1}^{\infty} a_k(t) s_k(x)$$

converges for all t in T and x in A and if

$$\lim_{t \to t_0} \sigma(t, x) = \sigma(x), \S$$

then (G) is said to assign the value $\sigma(x)$ to the sequence $\{s_n(x)\}$.

It is convenient to regard (G) as being a transformation which carries a *given sequence* $\{s_n(x)\}$ into a *transformed function* $\sigma(t, x)$. The transformation (G) is said to be real if $a_k(t)$ is real for all k and for all t in T; otherwise it is

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† These Transactions, vol. 32 (1930), pp. 669–708. This paper will be referred to hereafter as Trans.

‡ See Carmichael, Bulletin of the American Mathematical Society, vol. 25 (1918–19), p.118; J. Schur, Journal für Mathematik, vol. 151 (1920), p. 82; and W. A. Hurwitz, Bulletin of the American Mathematical Society, vol. 28 (1922), p. 18.

§ Here, as elsewhere in this paper, t is restricted to approaching t_0 over the set T.

complex. Except in cases where a specific statement to the contrary is made, the transformations and sequences considered in this paper may be complex. The following conditions* are listed together for convenience:

C_1: $\sum_{k=1}^{\infty} |a_k(t)|$ is bounded for all t in T;

C_2: for each k, $\lim_{t\to t_0} a_k(t) = 0$;

C_3: $\lim_{t\to t_0} \sum_{k=1}^{\infty} a_k(t) = 1$;

C_4: $\lim_{t\to t_0} \sum_{k=1}^{\infty} | a_k(t) | = 1$;

C_5: for each k, $a_k(t) = 0$ for all sufficiently advanced t;†

C_6: $\sum_{k=1}^{\infty} a_k(t) = 1$ for all sufficiently advanced t.

Since we are considering only *regular* transformations‡ we shall use the symbol (G) to represent a regular transformation. It is well known that C_1, C_2, and C_3 are necessary and sufficient for the regularity of a complex transformation when applied to complex sequences, and for the regularity of a real transformation when applied to real sequences. *Hence* (G), *complex or real, satisfies* C_1, C_2, *and* C_3.

3. Oscillations

Let a sequence $\{s_n(x)\}$ be defined over a set A. For continuous oscillations of a sequence we have the two following definitions. The continuous oscillation of $\{s_n(x)\}$ over the set A (which we shall call the *Ω-oscillation of $\{s_n(x)\}$ over A*) is denoted by $\Omega(\{s_n\}, A)$ and is defined as follows: for each sequence $\{x_i\}$ of points of A, form

$$\limsup_{m\to\infty, n\to\infty, i\to\infty, j\to\infty} | s_m(x_i) - s_n(x_j) | = v;$$

the least upper bound of all such v is $\Omega(\{s_n\}, A)$. Similarly the continuous

* These conditions are analogous to the corresponding conditions of Trans. To see this, we may specialize (G) as in §9 of this paper, and then impose the further condition $a_{nk}=0$, $k>n$. Then (G) assumes the form $\sigma_n(x)=\sum_{k=1}^{n} a_{nk}s_k(x)$, of a transformation with a triangular matrix; and the conditions $C_1, \cdots, C_6$ become the corresponding conditions of Trans. Owing to these circumstances, the lemmas and theorems of §§4–7 include corresponding results of Trans. to which the reader will be referred by footnotes.

† I.e. for all points t of T which lie in a sufficiently small neighborhood of t_0 in T.

‡ A transformation is said to be regular when it assigns to each convergent sequence the value to which it converges.

oscillation of $\{s_n(x)\}$ at a point x_0 of A^0* over the set A (which we shall call the *Ω-oscillation of* $\{s_n(x)\}$ *at* x_0 *over* A) is denoted by $\Omega(x_0; \{s_n\}, A)$ and is defined as follows: for each sequence $\{x_i\}$ of points of A with the limit x_0, form

$$\limsup_{m\to\infty,\, n\to\infty,\, i\to\infty,\, j\to\infty} |s_m(x_i) - s_n(x_j)| = v;$$

the least upper bound of all such v is $\Omega(x_0; \{s_n\}, A)$.

For uniform oscillations of sequences, we have the two following definitions. The uniform oscillation of $\{s_n(x)\}$ over the set A (which we shall call the *O-oscillation of* $\{s_n(x)\}$ *over* A) is denoted by $O(\{s_n\}, A)$ and is defined as follows: for each sequence $\{x_i\}$ of A, form

$$\limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} |s_m(x_i) - s_n(x_i)| = v;$$

the least upper bound of all such v is $O(\{s_n\}, A)$. Similarly the uniform oscillation of $\{s_n(x)\}$ at a point x_0 of A^0 over the set A (which we shall call the *O-oscillation of* $\{s_n(x)\}$ *at* x_0 *over* A) is denoted by $O(\{s_n\}, A)$ and is defined as follows: for each sequence $\{x_i\}$ of points of A with the limit x_0, form

$$\limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} |s_m(x_i) - s_n(x_i)| = v;$$

the least upper bound of all such v is $O(x_0; \{s_n\}, A)$.

For the corresponding Ω-oscillations of transformed functions, we have the two following definitions. Let t and u be points of T, and for each sequence $\{x_i\}$ of points of A, form

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} |\sigma(t, x_i) - \sigma(u, x_j)| = v;$$

the least upper bound of all such v is the *Ω-oscillation of* $\sigma(t, x)$ *over* A and will be denoted by $\Omega(\sigma, A)$. Similarly, for each sequence $\{x_i\}$ of A with the limit x_0, form

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} |\sigma(t, x_i) - \sigma(u, x_j)| = v;$$

the least upper bound of all such v is the *Ω-oscillation of* $\sigma(t, x)$ *at* x_0 *over* A and will be denoted by $\Omega(x_0; \sigma, A)$.

For the *O-oscillations* of transformed functions, we have the two following definitions. For each sequence $\{x_i\}$ of points of A, form

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |\sigma(t, x_i) - \sigma(u, x_i)| = v;$$

* A^0 is used to denote the set consisting of A and its limit points.

the least upper bound of all such v is the *O-oscillation of* $\sigma(t, x)$ *over* A and will be denoted by $O(\sigma, A)$. Similarly, for each sequence $\{x_i\}$ of points of A with the limit x_0, form

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} \left| \sigma(t, x_i) - \sigma(u, x_i) \right| = v;$$

the least upper bound of all such v is the *O-oscillation of* $\sigma(t, x)$ *at* x_0 *over* A and will be denoted by $O(x_0; \sigma, A)$.

4. Some fundamental lemmas

LEMMA 4.1. *If (G) fails to satisfy* C_4, *then there is a bounded sequence* $\{s_n\}$ *of constants such that*

$$\limsup_{t\to t_0,\, u\to t_0} \left| \sigma(t) - \sigma(u) \right| > \limsup_{m\to\infty,\, n\to\infty} \left| s_m - s_n \right| .$$

If (G) is real, $\{s_n\}$ *may be taken real.**

From C_3 and a denial of C_4, it follows that there is a number θ for which

$$\limsup_{t\to t_0} \sum_{k=1}^{\infty} \left| a_k(t) \right| > \theta > 1.$$

Hence there is a sequence $\{t_n\}$ with the limit t_0 such that

$$(4.11) \qquad \sum_{k=1}^{\infty} \left| a_k(t_n) \right| > \theta \qquad (n = 1, 2, 3, \cdots).$$

Let n_1 be any positive integer and choose $N_1 > n_1$ such that

$$\sum_{k=N_1+1}^{\infty} \left| a_k(t_{n_1}) \right| < 1.$$

Using C_2, choose $n_2 > N_1$ such that

$$\sum_{k=1}^{N_1} \left| a_k(t_{n_2}) \right| < \tfrac{1}{2};$$

then choose $N_2 > n_2$ such that

$$\sum_{k=N_2+1}^{\infty} \left| a_k(t_{n_2}) \right| < \tfrac{1}{2}.$$

Proceeding in this manner, we may define a sequence $N_0 = 0 < n_1 < N_1 < n_2 < N_2 < n_3 < \cdots$ such that for $p = 1, 2, 3, \cdots$

* Compare Trans. Lemma 4.01 of which the proof was given by W. A. Hurwitz, American Journal of Mathematics, vol. 52 (1930), pp. 611–616.

$$(4.12)\qquad \sum_{k=1}^{N_{p-1}} |a_k(t_{n_p})| < \frac{1}{p} \quad\text{and}\quad \sum_{k=N_p+1}^{\infty} |a_k(t_{n_p})| < \frac{1}{p}\cdot$$

From these inequalities and (4.11) we obtain

$$(4.13)\qquad \sum_{k=N_{p-1}+1}^{N_p} |a_k(t_{n_p})| > \theta - \frac{2}{p}\cdot$$

Now define* for $p=1, 2, 3, \cdots$

$$(4.14)\qquad s_k = (-1)^{p+1}\operatorname{sgn} a_k(t_{n_p}), \quad N_{p-1} < k \leqq N_p.$$

Then s_k is real for all k if (G) is real; and $|s_k| \leqq 1$ for all k so that

$$\limsup_{m\to\infty,\, n\to\infty} |s_m - s_n| \leqq 2.$$

But

$$\sigma(t_{n_{p+1}}) - \sigma(t_{n_p}) = \sum_{k=1}^{\infty} a_k(t_{n_{p+1}})s_k - \sum_{k=1}^{\infty} a_k(t_{n_p})s_k,$$

and using (4.14) we obtain

$$\sigma(t_{n_{p+1}}) - \sigma(t_{n_p}) = \sum_{k=1}^{N_p} a_k(t_{n_{p+1}})s_k - \sum_{k=1}^{N_{p-1}} a_k(t_{n_p})s_k + \sum_{k=N_{p+1}+1}^{\infty} a_k(t_{n_{p+1}})s_k$$
$$- \sum_{k=N_p+1}^{\infty} a_k(t_{n_p})s_k + (-1)^p \sum_{k=N_p+1}^{N_{p+1}} |a_k(t_{n_{p+1}})| + (-1)^p \sum_{k=N_{p-1}+1}^{N_p} |a_k(t_{n_p})|\ .$$

Using (4.12) and the fact that $|s_k| \leqq 1$ for all k, we see that the sum of the absolute values of the first four terms of the right member of the last expression is less than

$$\frac{1}{p+1} + \frac{1}{p} + \frac{1}{p+1} + \frac{1}{p} < \frac{4}{p};$$

and using (4.13) we see that the absolute value of the sum of the last two terms (which are real and of like sign) is greater than

$$\left(\theta - \frac{2}{p+1}\right) + \left(\theta - \frac{2}{p}\right) > 2\theta - \frac{4}{p}\cdot$$

Hence

$$|\sigma(t_{n_{p+1}}) - \sigma(t_{n_p})| > 2\theta - \frac{8}{p}\cdot$$

Thus

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* For complex z, $\operatorname{sgn} z = |z|/z$ when $z \neq 0$ and $=0$ when $z=0$.

$$\limsup_{t\to t_0,\, u\to t_0} |\sigma(t) - \sigma(u)| \geqq \limsup_{p\to\infty} |\sigma(t_{n_{p+1}}) - \sigma(t_{n_p})| \geqq \lim_{p\to\infty}\left(2\theta - \frac{8}{p}\right)$$
$$= 2\theta > 2 \geqq \limsup_{m\to\infty,\, n\to\infty} |s_m - s_n|,$$

and the lemma is proved.

The four following lemmas may be proved together.

LEMMAS 4.2, 4.3. *Let A be an infinite set and let $\{x_\alpha\}$ be a sequence of distinct points of A. In case A has a limit point x_0, $\{x_\alpha\}$ may be a sequence with the limit x_0. If (G), real or complex, fails to satisfy* C_5, *then there is a real sequence $\{s_n(x)\}$ bounded above (below) over A, such that*

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} |\sigma(t, x_i) - \sigma(u, x_j)| > \limsup_{m\to\infty,\, n\to\infty,\, i\to\infty,\, j\to\infty} |s_m(x_i') - s_n(x_j')| = 0$$

*where $\{x_\alpha'\}$ is any sequence of points of A.**

LEMMAS 4.4, 4.5. *Under the hypotheses of Lemmas* 4.2, 4.3, *there is a real sequence $\{s_n(x)\}$, bounded above (below) over A such that*

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |\sigma(t, x_i) - \sigma(u, x_i)| > \limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} |s_m(x_i') - s_n(x_i')| = 0,$$

where $\{x_\alpha'\}$ is any sequence of points of A.†

From a denial of C_5 it follows that there is a value of k, say λ, and a sequence $\{t_\alpha\}$ with the limit t_0 such that

$$a_\lambda(t_\alpha) \neq 0 \qquad (\alpha = 1, 2, 3, \cdots).$$

Define the sequence $\{s_n(x)\}$ over A as follows: $s_n(x)=0$ over A for $n\neq\lambda$; $s_\lambda(x)=0$, $x\neq x_1, x_2, \cdots$; and $s_\lambda(x_\alpha)=(-1)^h/|a_\lambda(t_\alpha)|$ where $h=1(h=2)$. Evidently $s_n(x)$ is bounded above or below over A according as h is 1 or 2, and since $s_n(x)=0$ over A for $n>\lambda$,

$$\limsup_{m\to\infty,\, n\to\infty,\, i\to\infty,\, j\to\infty} |s_m(x_i') - s_n(x_j')| = \limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} |s_m(x_i') - s_n(x_i')| = 0$$

where $\{x_\alpha'\}$ is any sequence of points of A. But $\sigma(t, x) = a_\lambda(t)s_\lambda(x)$ so that $|\sigma(t_\alpha, x_\alpha)| = 1$ and $|\sigma(t_\beta, x_\alpha)| = |a_\lambda(t_\beta)|/|a_\lambda(t_\alpha)|$; hence

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} \Big|\sigma(t, x_i) - \sigma(u, x_j)\Big| \geqq \limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} \Big|\sigma(t, x_i) - \sigma(u, x_i)\Big|$$
$$\geqq \limsup_{\alpha\to\infty,\, \beta\to\infty} \Big|\sigma(t_\alpha, x_\alpha) - \sigma(t_\beta, x_\alpha)\Big| \geqq \limsup_{\alpha\to\infty,\, \beta\to\infty} \Big||\sigma(t_\alpha, x_\alpha)| - |\sigma(t_\beta, x_\alpha)|\Big|$$
$$\geqq \limsup_{\alpha\to\infty,\, \beta\to\infty} \Big|1 - |a_\lambda(t_\beta)|/|a_\lambda(t_\alpha)|\Big| = +\infty,$$

* Compare Trans. Lemmas 4.02, 4.03.

† Compare Trans. Lemmas 4.04, 4.05.

and the lemmas are proved.

LEMMAS 4.6, 4.7. *Let A be an infinite set and let $\{x_\alpha\}$ be a sequence of distinct points of A. In case A has a limit point x_0, $\{x_\alpha\}$ may be a sequence with the limit x_0. If (G), real or complex, fails to satisfy* C_6, *then there is a real sequence $\{s_n(x)\}$, bounded above (below) over A, such that*

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |\sigma(t, x_i) - \sigma(u, x_i)| > \limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} |s_m(x_i') - s_n(x_i')| = 0$$

*where $\{x_i'\}$ is any sequence of points of A.**

From a denial of C_6 it follows that there is a sequence $\{t_\alpha\}$ with the limit t_0 such that

$$\sum_{k=1}^{\infty} a_k(t_\alpha) \neq 1 \qquad (\alpha = 1, 2, 3, \cdots).$$

Define a real function $s(x)$ over A, bounded above (below) over A, such that

$$s(x_\alpha) = (-1)^k \Big/ \left| 1 - \sum_{k=1}^{\infty} a_k(t_\alpha) \right| ;$$

and let $s_n(x) = s(x)$, $n = 1, 2, 3, \cdots$. Then $s_m(x) - s_n(x) = 0$ over A so that

$$\limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} |s_m(x_i') - s_n(x_i')| = 0$$

where $\{x_i'\}$ is any sequence of points of A. But

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |\sigma(t, x_i) - \sigma(u, x_i)| = \limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |s(x_i)| \left| \sum_{k=1}^{\infty} a_k(t) - \sum_{k=1}^{\infty} a_k(u) \right|$$

$$\geqq \limsup_{u\to t_0,\, i\to\infty} |s(x_i)| \left| 1 - \sum_{k=1}^{\infty} a_k(u) \right| \geqq \limsup_{\alpha\to\infty} |s(x_\alpha)| \left| 1 - \sum_{k=1}^{\infty} a_k(t_\alpha) \right| = 1$$

and the lemmas are proved.

5. PROOFS OF TYPICAL THEOREMS INVOLVING Ω-OSCILLATIONS OVER A SET

THEOREM 5.11. *In order that (G) may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, defined over an arbitrary set A and bounded over A for all n, C_4 *is necessary and sufficient.*†

The necessity of C_4 follows from Lemma 4.1 since, for the sequence of constants there defined, $\Omega(\sigma, A) > \Omega(\{s_n\}, A)$. To establish sufficiency of C_4,

* Compare Trans. Lemmas 7.01, 7.02.

† Compare Trans. Theorem 4.111.

choose M such that $|s_n(x)| < M$ over A for all n, B such that $\sum_{k=1}^{\infty}|a_k(t)| < B$ for all t in T and let $\sum_{k=1}^{\infty}|a_k(t)| = B(t)$. Let $\{x_i\}$ be any sequence of points of A, and let q be any number greater than $\Omega(\{s_n\}, A)$; then there is an index p such that

$$(5.111)\qquad |s_\mu(x_i) - s_\nu(x_j)| < q \text{ for } \mu \geqq p,\ \nu \geqq p,\ i \geqq p,\ j \geqq p.$$

We readily obtain the identity

$$(5.112)\quad \begin{aligned} \sigma(t, x_i) - \sigma(u, x_j) &= \sum_{k=1}^{p} a_k(t)s_k(x_i) - \sum_{k=1}^{p} a_k(u)s_k(x_j) \\ &+ \left(\sum_{\mu=p+1}^{\infty} a_\mu(t)s_\mu(x_i)\right)\left(1 - \sum_{\nu=p+1}^{\infty} a_\nu(u)\right) - \left(\sum_{\nu=p+1}^{\infty} a_\nu(u)s_\nu(x_j)\right)\left(1 - \sum_{\mu=p+1}^{\infty} a_\mu(t)\right) \\ &+ \sum_{\mu=p+1}^{\infty}\sum_{\nu=p+1}^{\infty} a_\mu(t)a_\nu(u)\left[s_\mu(x_i) - s_\nu(x_j)\right]. \end{aligned}$$

The absolute values of the first four terms of the right member of (5.112) are respectively less than or equal to

$$M\sum_{k=1}^{p}|a_k(t)|,\quad M\sum_{k=1}^{p}|a_k(u)|,\quad MB\left|1 - \sum_{\nu=p+1}^{\infty} a_\nu(u)\right|,\quad \text{and}\quad MB\left|1 - \sum_{\mu=p+1}^{\infty} a_\mu(t)\right|,$$

each of which, by C_2 and C_3, approaches 0 as t and u approach t_0. Hence

$$\begin{aligned} &\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} |\sigma(t, x_i) - \sigma(u, x_j)| \\ &= \limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} \left|\sum_{\mu=p+1}^{\infty}\sum_{\nu=p+1}^{\infty} a_\mu(t)a_\nu(u)\left[s_\mu(x_i) - s_\nu(x_j)\right]\right|. \end{aligned}$$

But by (5.111)

$$\begin{aligned} \left|\sum_{\mu=p+1}^{\infty}\sum_{\nu=p+1}^{\infty} a_\mu(t)a_\nu(u)\left[s_\mu(x_i) - s_\nu(x_j)\right]\right| &\leqq q\sum_{\mu=p+1}^{\infty}\sum_{\nu=p+1}^{\infty}|a_\mu(t)|\,|a_\nu(u)| \\ &\leqq qB(t)B(u) \end{aligned}$$

so that

$$(5.113)\qquad \limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} |\sigma(t, x_i) - \sigma(u, x_j)| \leqq \limsup_{t\to t_0,\, u\to t_0} \left[qB(t)B(u)\right];$$

and using C_4 we have

$$(5.114)\qquad \limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} |\sigma(t, x_i) - \sigma(u, x_j)| \leqq q.$$

Since $\{x_i\}$ is any sequence of points of A, it follows from (5.114) that

$\Omega(\sigma, A) \leqq q$; and since q is any number greater than $\Omega(\{s_n\}, A)$, $\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$ and the theorem is proved.

On noting that the proof of the preceding theorem is undisturbed by supposing (G) and the $\{s_n(x)\}$ sequences to be real, we obtain

THEOREM 5.12. *In order that a real* (G) *may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every real sequence $\{s_n(x)\}$ *defined over an arbitrary set* A, *and bounded over* A *for all* n, C_4 *is necessary and sufficient.**

THEOREM 5.13. *In order that* (G) *may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, *defined over an infinite set* A, C_4 *and* C_5 *are necessary and sufficient.*†

Necessity of C_4 follows from Lemma 4.1; and that of C_5 from Lemma 4.2 (or 4.3) since for the $\{s_n(x)\}$ sequences there defined, $\Omega(\{s_n\}, A) = 0$ while $\Omega(\sigma, A) > 0$. No proof of sufficiency is required if $\Omega(\{s_n\}, A) = +\infty$. If $\Omega(\{s_n\}, A)$ is finite, let q be any greater number, let $\{x_i\}$ be any sequence of points of A, and choose an index p such that (5.111) is satisfied. Then $|s_n(x_i) - s_p(x_p)| < q$ for $n \geqq p$, $i \geqq p$; hence there is a constant M such that $|s_n(x_i)| < M$ for $n \geqq p$, $i \geqq p$. Using C_5, choose a neighborhood Δ of t_0 in T such that $a_k(t) = 0$, $k = 1, 2, 3, \cdots, p$ for t in Δ. Then, referring to the identity (5.112), we see that the first two terms of the right member vanish for t and u in Δ and that, for $i \geqq p, j \geqq p$, the second and third terms approach zero as t and u approach t_0. Therefore we may write (5.113) and sufficiency follows as in Theorem 5.11. The same proof establishes the following two theorems.

THEOREM 5.14. *In order that a real* (G) *may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every real sequence $\{s_n(x)\}$, *defined over an infinite set* A, C_4 *and* C_5 *are necessary and sufficient.*‡

THEOREM 5.15. *In order that a real* (G) *may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every real sequence $\{s_n(x)\}$, *defined over an infinite set* A *and bounded above (below) over* A *for all* n, C_4 *and* C_5 *are necessary and sufficient.*§

* Compare Trans. Theorem 4.112.

† Compare Trans. Theorem 4.131.

‡ Compare Trans. Theorem 4.132.

§ Compare Trans. Theorem 4.133.

THEOREM 5.21. *In order that (G) may be such that $\Omega(\sigma, A)=0$ for every sequence $\{s_n(x)\}$, defined over an arbitrary set A and bounded over A for all n, such that $\Omega(\{s_n\}, A)=0$, no further conditions need be imposed upon a_{nk}.**

Letting $\{x_i\}$ be any sequence of points of A, and q be an arbitrarily small positive number, we can choose an index p for which (5.111) holds; and using (5.112) obtain (5.113) precisely as in Theorem 5.11. But

$$\limsup_{t\to t_0,\, u\to t_0} qB(t)B(u) \leqq qB^2;$$

hence $\Omega(\sigma, A)\leqq qB^2$. Since qB^2 is arbitrarily small, $\Omega(\sigma, A)=0$ and the theorem is proved.

THEOREM 5.22. *In order that (G) may be such that $\Omega(\sigma, A)=0$ for every sequence $\{s_n(x)\}$, defined over an infinite set A, such that $\Omega(\{s_n\}, A)=0$, C_5 is necessary and sufficient.*†

Necessity follows from Lemma 4.2 (or 4.3). The sufficiency proof is a modification of that of Theorem 5.13 in the same sense that the proof of Theorem 5.21 is a modification of the sufficiency proof of Theorem 5.11. The same proof establishes the two following theorems.

THEOREM 5.23. *In order that a real (G) may be such that $\Omega(\sigma, A)=0$ for every real sequence $\{s_n(x)\}$, defined over an infinite set A, such that $\Omega(\{s_n\}, A)=0$, C_5 is necessary and sufficient.*‡

THEOREM 5.24. *In order that a real (G) may be such that $\Omega(\sigma, A)=0$ for every real sequence $\{s_n(x)\}$, defined over an infinite set A and bounded above (below) over A for all n, such that $\Omega(\{s_n\}, A)=0$, C_5 is necessary and sufficient.*§

6. PROOF OF A TYPICAL THEOREM INVOLVING Ω-OSCILLATIONS AT A POINT OVER A SET

THEOREM 6.1. *In order that (G) may be such that*

$$\Omega(x_0;\, \sigma, A) \leqq \Omega(x_0;\, \{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, defined over a set A such that x_0 is in A^0 and bounded over a neighborhood D of x_0 in A for all n, C_4 is necessary and sufficient.‖

Necessity follows from Lemma 4.1. To establish sufficiency, choose M such that $|s_n(x)|<M$ over D for all n, let $\{x_i\}$ be any sequence of points of

* Compare Trans. Theorem 4.22.

† Compare Trans. Theorem 4.231.

‡ Compare Trans. Theorem 4.232.

§ Compare Trans. Theorem 4.233.

‖ Compare Trans. Theorem 5.111.

A with the limit x_0, and let q be any number greater than $\Omega(x_0;\ \{s_n\},\ A)$. Then there is an index p for which (5.111) holds; let p be increased if necessary so that x_i is in D for $i \geqq p$. Using (5.112), we obtain (5.114) precisely as in Theorem 5.11; therefore $\Omega(x_0,\ \sigma,\ A) \leqq q$ and sufficiency follows.

7. Proof of a typical theorem involving O-oscillations over a set

THEOREM 7.1. *In order that* (G) *may be such that*

$$O(\sigma,\ A) \leqq O(\{s_n\},\ A)$$

for every sequence $\{s_n(x)\}$, *defined over an infinite set* A, C_4, C_5, *and* C_6 *are necessary and sufficient.**

Necessity of C_4 follows from Lemma 4.1; and that of C_5 and C_6 from Lemmas 4.4 (or 4.5) and 4.6 (or 4.7) respectively, since, for the $\{s_n(x)\}$ sequences there defined, $O(\{s_n\},\ A)=0$ while $O(\sigma,\ A)>0$. If $O(\{s_n\},\ A) = +\infty$, no proof of sufficiency is required. If $O(\{s_n\}, A)$ is finite, let q be any greater number and let $\{x_i\}$ be any sequence of points of A; then there is an index p such that $|s_\mu(x_i)-s_\nu(x_i)|<q$ for $\mu \geqq p$, $\nu \geqq p$, $i \geqq p$. Using C_5 and C_6, choose a neighborhood Δ of t_0 in T such that $a_k(t)=0$, $k=1, 2, 3, \cdots, p$, for t in Δ and also $\sum_{k=p+1}^{\infty} a_k(t)=1$ for t in Δ. Then, considering the identity (5.112) with j replaced by i, we see that the first four terms of the right member vanish for t and u in Δ; hence we obtain (5.113) and, using C_4, (5.114) with j replaced by i. Therefore $O(\sigma, A) \leqq q$, $O(\sigma, A) \leqq O(\{s_n\}, A)$, and the theorem is proved.

8. A catalogue of theorems

A comparison of the proofs which have been given in §§5–7 with those of the corresponding theorems of Trans. will suggest to the reader all of the modifications of the proofs of the theorems of Chapters II and III of Trans. which are necessary to obtain new theorems involving the general regular transformation (G). We shall, to save space, not give formal statements of the new theorems but shall specify the changes which must be made in the theorems of Trans. to produce the new theorems.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *Ω-oscillation over a set of a transformed function shall not exceed the Ω-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $\Omega(\{\sigma_n\}, A)$ by $\Omega(\sigma, A)$ in Trans. Theorems 4.111–4.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *Ω-oscillation over a set of a transformed function shall be zero whenever the Ω-oscillation over the set of a given sequence is zero*, is obtained

* Compare Trans. Theorem 7.131.

by replacing (T) by (G) and $\Omega(\{\sigma_n\}, A)$ by $\Omega(\sigma, A)$ in Trans. Theorems 4.21–4.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at a single point or limit point of a set, the Ω-oscillation over the set of a transformed function shall not exceed the Ω-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $\Omega(x_0; \{\sigma_n\}, A)$ by $\Omega(x_0, \sigma, A)$ in Trans. Theorems 5.111–5.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at a single point or limit point of a set, the Ω-oscillation over the set of a transformed function shall be zero whenever the Ω-oscillation over the set of a given sequence is zero*, is obtained by replacing (T) by (G) and $\Omega(x_0; \{\sigma_n\}, A)$ by $\Omega(x_0, \sigma, A)$ in Trans. Theorems 5.21–5.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at each point and limit point of a set, the Ω-oscillation over the set of a transformed function shall not exceed the Ω-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $\Omega(x; \{\sigma_n\}, A)$ by $\Omega(x, \sigma, A)$ in Trans. Theorems 6.111–6.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *the Ω-oscillation of a transformed function shall be zero at each point and limit point of a set whenever the Ω-oscillation of a given sequence is zero at each point and limit point of the set*, is obtained by replacing (T) by (G) and $\Omega(x; \{\sigma_n\}, A)$ by $\Omega(x, \sigma, A)$ in Trans. Theorems 6.21–6.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *O-oscillation over a set of a transformed function shall not exceed the O-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $O(\{\sigma_n\}, A)$ by $O(\sigma, A)$ in Trans. Theorems 7.111–7.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *O-oscillation over a set of a transformed function shall be zero whenever the O-oscillation over the set of a given sequence is zero*, is obtained by replacing (T) by (G) and $O(\{\sigma_n\}, A)$ by $O(\sigma, A)$ in Trans. Theorems 7.21–7.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at a single point or limit point of a set, the O-oscillation over the set of a transformed function shall not exceed the O-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $O(x_0; \{\sigma_n\}, A)$ by $O(x_0, \sigma, A)$ in Trans. Theorems 8.111–8.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at a single point or limit point of a set, the O-oscillation over the set of a transformed function shall be zero whenever the O-oscillation over the* *set of a given sequence is zero*, is obtained by replacing (T) by (G) and $O(x_0; \{\sigma_n\}, A)$ by $O(x_0, \sigma, A)$ in Trans. Theorems 8.21–8.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at each point and limit point of a set, the O-oscillation over the set of a transformed function shall not exceed the O-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $O(x; \{\sigma_n\}, A)$ by $O(x, \sigma, A)$ in Trans. Theorems 9.111–9.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *O-oscillation of a transformed function shall be zero at each point and limit point of a set whenever the O-oscillation of a given sequence is zero at each point and limit point of the set*, is obtained by replacing (T) by (G) and $O(x; \{\sigma_n\}, A)$ by $O(x, \sigma, A)$ in Trans. Theorems 9.21–9.233.

9. Application to transformations with square matrices

A well known family of regular transformations of the form (G) is obtained by taking T to be the set of positive integers and t_0 to be the symbolic limit point $+\infty$. Then $a_k(n)$ and $\sigma(n, x)$ may be written a_{nk} and $\sigma_n(x)$, and (G) becomes a transformation of the form

$$(S) \qquad \sigma_n(x) = \sum_{k=1}^{\infty} a_{nk} s_k(x)$$

which assigns to the sequence $\{s_n(x)\}$ the value $\lim_{n\to\infty}\sigma_n(x) = \sigma(x)$ when the limit exists. In this case what we have called the *transformed function* becomes a *transformed sequence* and we see on referring to the definitions of §3 that the oscillations $\Omega(\sigma, A)$, $\Omega(x_0, \sigma, A)$, $O(\sigma, A)$, and $O(x_0, \sigma, A)$ become respectively $\Omega(\{\sigma_n\}, A)$, $\Omega(x_0; \{\sigma_n\}, A)$, $O(\{\sigma_n\}, A)$, and $O(x_0; \{\sigma_n\}, A)$. Hence for regular transformations (S), the statements of the theorems of §8 become practically identical with the statements of the corresponding theorems of Chapters II and III of Trans. In fact, we may obtain, from each theorem of Chapters II and III of Trans., a theorem involving (S) by replacing (T) by (S) and interpreting C_4, C_5, and C_6 to be the conditions obtained by replacing t by n, $a_k(t)$ by a_{nk}, and $\lim_{t\to t_0}$ by $\lim_{n\to\infty}$ in the conditions of §2.

10. Application of the Euler-Abel power series method

The Euler-Abel transformation assigns to a series $u_1 + u_2 + \cdots$ the value

$$\lim_{t\to 1} \sigma(t) = \lim_{t\to 1} (u_1 + u_2 t + u_3 t^2 + \cdots)$$

when the limit exists, and to a sequence $\{s_n\}$ the value $\lim_{t\to 1}\sigma(t)$, where

$$(E) \qquad \sigma(t) = \sum_{k=1}^{\infty} t^{k-1}(1 - t)s_k,$$

when the limit exists. That the transformation (E) is regular when the set T over which t approaches 1 is the real set $-1<t<1$ was first shown by Abel. It has been pointed out by Hurwitz* that a necessary and sufficient condition that (E) be regular is that the set T be a set of a region R interior to the circle $|t|=1$ and between some pair of chords through $t=1$.

For (E) we have, in the notation of (G), $a_k(t)=t^{k-1}(1-t)$; and we see at once that (E) *fails to satisfy* C_5 *when* T *is any set of* R, *and that* (E) *satisfies* C_6 *when* T *is any set of* R.

We find further: *In order that a regular* (E) *may satisfy* C_4, *it is necessary and sufficient that* T *be such that for each sequence* $t_n=\xi_n+i\eta_n$ *of points of* T *with the limit* 1, *we have* $\lim_{n\to\infty}\eta_n/(1-\xi_n)=0$. In particular, *if* T *is the set* $-1<t<1$, *then* (E) *satisfies* C_4.

11. Application of the Borel-Sannia transformations

For each integer r (positive, zero, and negative), the Borel-Sannia transformation of order r† is given by

$$(B_r) \qquad \sigma^{(r)}(t) = \sum_{k=1}^{\infty} e^{-t}\frac{t^{k-r}}{(k-r)!}s_k,‡$$

and it assigns to a sequence $\{s_n\}$ the value $\lim_{R(t)\to+\infty}\sigma^{(r)}(t)$ when this limit exists. A necessary and sufficient condition that (B_r) be regular is that for all points $t=\xi+i\eta$ of T with sufficiently great positive abscissas, η^2/ξ shall be bounded.§

For (B_r), $a_k^{(r)}(t)=e^{-t}t^{k-r}/(k-r)!$ and we see that *any regular* (B_r) *fails to satisfy* C_5. Considering C_6, we find that *any regular* (B_r) *of order* $r\geqq 1$ *satisfies* C_6 *and any regular* (B_r) *of order* $r\leqq 0$ *fails to satisfy* C_6.

For each integer r we find the following: *In order that a regular* (B_r) *may satisfy* C_4, *it is necessary and sufficient that* T *be such that for each sequence* $t_n=\xi_n+i\eta_n$ *of points of* T *such that* $\xi_n\to+\infty$, *we have* $\lim\eta_n^2/\xi_n=0$. In particular, *if* T *is the set of positive real numbers, then* (B_r) *satisfies* C_4.

* Bulletin of the American Mathematical Society, vol. 28 (1922), p. 24.

† G. Sannia, Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 303–322. Note that B_1 is the Borel mean or the Borel exponential transformation, and that B_0 is the Borel integral transformation.

‡ Here $1/(k-r)!=0$ where $k-r<0$. This convention is justified by the behavior of the reciprocal of the gamma function.

§ W. A. Hurwitz, Bulletin of the American Mathematical Society, loc. cit., p. 25.

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